Some Abstract Algebra - A Primer

and some Number Theory

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Contents

DAY 1 PROPERTIES OF THREE TABLES

$\bullet$ = usual complex multiplication

$\bullet$	1	-1	i	$-i$
1				
-1				
i				
$-i$				

PROPERTIES, OBSERVATIONS

1.

2.

3.

$\otimes$ = units digit in regular multiplication

$\otimes$	1	3	7	9
1				
3				
7				
9				

1.

2.

3.

$\oplus$ = bitwise addition, 0 if same, 1 if different

$\oplus$	00	01	10	11
00				
01				
10				
11				

1.

2.

3.

WORKSHEET ON CLOSURE I

A set $S = \{a, b, c, \dots\}$ is closed under a binary operation $\circ$ if whenever x and y are elements of S so is $x \circ y$.

For each of the following if the answer is yes, give a reason and if no, provide a counterexample.

Task 1 Is $E = \{0, 2, 4, 6, 8, \dots\}$ closed under the binary operation of addition?

☐ yes, ☐ no   Reason: Let $2m$ and $2n$ be arbitrary elements in E.

Then since ...

How about under multiplication?

Task 2 Is $A = \{0, 1, 4, 9, 16, \dots\}$ closed under addition?

Under subtraction?

Under multiplication?

Task 3 Is the set of all rational numbers of the form $2^m 3^n$, where $m, n \in \mathbb{Z}$, closed under multiplication?

Task 4 Is the set of all positive rational numbers closed under addition? Multiplication?

Task 5 Are the complex numbers of the form $m + ni$ where m and n are integers closed under multiplication?

Task 6 Is the set $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ closed under multiplication?

Task 7 Are the irrationals closed under multiplication? Under subtraction?

WORKSHEET ON CLOSURE II

Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
QUESTION: Which of the sets $3\mathbb{Z}$, $1 + 3\mathbb{Z}$, $2 + 3\mathbb{Z}$ are closed under subtraction?

Task 1 List the elements of $3\mathbb{Z}$; choose two and subtract them.

Task 2 What does it mean to say $3\mathbb{Z}$ is closed under subtraction?

Task 3 Is $3\mathbb{Z}$ closed under subtraction? If yes, prove it.

Task 4 Is $1 + 3\mathbb{Z}$ closed under subtraction?

Task 5 Is $2 + 3\mathbb{Z}$ closed under subtraction?

Task 6 Why must a set of integers contain 0 to be closed under subtraction?

WORKSHEET ON CLOSURE III

PROBLEM: Prove that if S and T are sets of integers closed under subtraction so is the intersection $S \cap T$.

Task 1 Say in your own words what it means to say S is closed under subtraction.

Task 2 What do you have to show in order to check that $S \cap T$ is closed under subtraction?

Task 3 Draw a Venn diagram as an aid, and resolve the problem.

Task 4 If S and T are sets of integers closed under subtraction is the union $S \cup T$ also closed under subtraction? If yes, prove it, if no give a counterexample.

WORKSHEET ON RESIDUE CLASSES

Congruence Modulo m is an EQUIVALENCE RELATION on $\mathbb{Z}$, the set of all integers.

R − REFLEXIVE: $a \equiv a(m)$
S − SYMMETRIC: If $a \equiv b(m)$ then $b \equiv a(m)$
T − TRANSITIVE: $a \equiv b(m)$ and $b \equiv c(m)$ then $a \equiv c(m)$

The relation *congruence* partitions $\mathbb{Z}$ into disjoint EQUIVALENCE CLASSES or RESIDUE CLASSES.

When $m = 2$, $\mathbb{Z}$ is partitioned into the classes $2\mathbb{Z}$ and $1 + 2\mathbb{Z}$.

$$2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$
$$1 + 2\mathbb{Z} = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$$

Task 1 Explain why the classes $2\mathbb{Z}$ and $1 + 2\mathbb{Z}$ are disjoint.

Task 2 What the residue classes when $m = 3$? Are they disjoint? Why?

Task 3 When $m = 4$? Explain.

WORKSHEET ON MODULAR ARITHMETIC

$a \equiv b (\text{mod } m)$ means a and b have the same remainder when divided by m; or that $a - b$ is divisible by m, or $a - b = mk$. An example: $17 \equiv 9 (\text{mod } 4)$ since 4 divides $17 - 9$.

Task 1 Complete the missing four rows:

	0	1	2	3	4	5	6	7	8	9	10	11	12
Mod 2													
Mod 3	0	1	2	0	1	2	0	1	2	0	1	2	0
Mod 4													
Mod 5													
Mod 6													

Task 2 Tables of addition Mod 5 and Mod 6 would look like:

$\oplus$	0	1	2	3	4
0					
1	1	2	3	4	0
2					
3					
4					

$\oplus$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

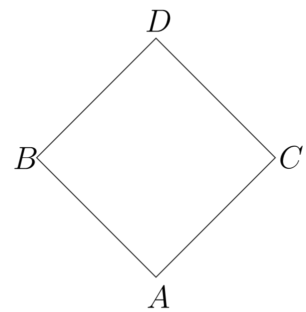
Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be the six elements you used to make the 6 by 6 table in Task 2. If you examine that addition table, you can see that each of the following subsets are closed under the binary operation $\oplus$. The relationship among these subsets is shown in the diagram.

$A = \{0\}$

$B = \{0, 3\}$

$C = \{0, 2, 4\}$

$D = \{0, 1, 2, 3, 4, 5\}$



Task 3 Make the $\oplus$ table for $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, list all the subsets closed under $\oplus$, and make a diagram as above.

$\oplus$	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

$A = \{0\}$

$B =$

$C =$

$D =$

Task 4 Without making the addition table, can you give all the closed subsets of $\mathbb{Z}_{12}$?

WORKSHEET ON CANCELLATION

Cancellation Theorem: If either $ab = ac$ or $ba = ca$ in a group G, then $b = c$.

Task 1 Let's try to prove right cancellation.

The hypothesis for right cancellation is:

If the element a is in G, __________________ is also in G.

Now show how to use this latter element on $b\,a = c\,a$ and conclude that $b = c$.

– CONNECTIONS –

Task 2 Let A, B, C be sets in a universe S. If $A \cup B = A \cup C$ is it necessarily true that $B = C$?

Task 3 Does $A \cap B = A \cap C$ imply $B = C$?

Task 4 For 2 by 2 matrices A, B, C does $AB = AC$ imply $B = C$?

Task 5 For real numbers x, y, z does $x + y = x + z$ imply $y = z$?

PERMUTATIONS

Each permutation on $X_4 = \{1, 2, 3, 4\}$ is a 1–1, onto function f. For example, the permutation $1 \to 2$, $2 \to 4$, $3 \to 3$, $4 \to 1$ has the function table

x	1	2	3	4
$f(x)$	2	4	3	1

and can be expressed in cycle form as (124). With "multiplication" being composition of functions the product (124)(23) is (1324), operating left to right. The cycle form (124) means $1 \to 2 \to 4 \to 1$ with 3 fixed. The product (124)(23) can be written as two-rowed arrays as:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (1324)$$

We list cycles in standard form as follows:

1. Smallest number first
2. Omission of a number m means $m \to m$ is fixed

Task 1 a=(1342)

$a^2 =$

$a^3 =$

$a^4 =$

a=(24675)

$a^2 =$

$a^3 =$

$a^4 =$

$a^5 =$

Task 2 If β =(26), $\beta^{-1} =$

What is the inverse of any transposition (ab)?____________________________

Task 3 Let α=(132)(4675). What is the smallest positive integer s so that

$\alpha^s = (1)$? s=________________________.

Repeat with β=(12)(3465): s=________________________.

Give the order of each element by filling in the chart:

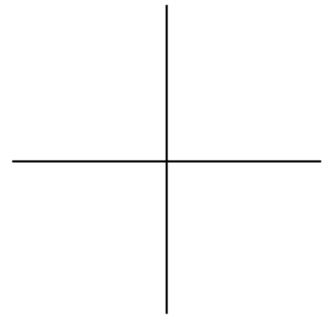
α	(13)	(132)	(12)(34)	(1432)	(132)(23)	(13)(12)
order α						

What is the order of $\beta = (13)(257)(4689)$?

What is the order of $\gamma = (13)(234)$?

WORKSHEET ON SUBGROUPS

Let $H = \{\alpha : \alpha = a + bi, |\alpha| \leq 1\}$. Is H a subgroup of the multiplicative group of non-zero complex numbers?



Task 1 Draw a picture showing all α with $|\alpha| \leq 1$.
Recall $|a + bi| = \sqrt{a^2 + b^2}$.

Task 2 To show that H is a subgroup we need to show, for one thing, that if $\alpha \in H$ so is α^{-1}. What is the inverse of $a + bi$? Try this on $\alpha = \frac{1}{2} + \frac{1}{2}i$. What is α^{-1}?

Is $\alpha \in H$? You need to compute $\sqrt{\frac{1}{4} + \frac{1}{4}}$.

Draw α and α^{-1} in your picture as vectors. How are their angles related? How do you multiply two complex numbers to show $\alpha\alpha^{-1} = 1$?

Task 3 Give an α not in H.

Task 4 Do you now need to check closure?

WORKSHEET ON ORDER

The order of an element a of a group G is the order of the cyclic subgroup $[a]$ generated by a in G. Equivalently, it is the smallest positive integer m so that $a^m = e$.

Task 1 Let G be a cyclic group of order 18 generated by a. Then

$$G = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, a^{16}, a^{17}\}.$$

Give the order of each element of G by filling out the chart:

x	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
order x									

x	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}
order x									

Task 2 Repeat Task 1 with G being a cyclic group of order 24.

WORKSHEET ON GROUP TABLES

Give as many reasons as you can why each of these tables cannot be group operation tables. You can state a group axiom that fails, or appeal to some of our theorems and results.

	a	b	c	d	e
a	c	e	a	b	d
b	d	c	b	e	a
c	a	b	c	d	e
d	e	a	d	c	b
e	b	d	e	a	c

	a	b	c	d	e
a	c	e	a	b	d
b	d	a	b	e	c
c	a	b	c	d	e
d	e	c	d	a	b
e	b	d	e	c	a

	a	b	c	d	e
a	e	d	b	c	a
b	c	e	d	a	b
c	d	a	e	b	c
d	b	c	a	e	d
e	a	b	c	d	e

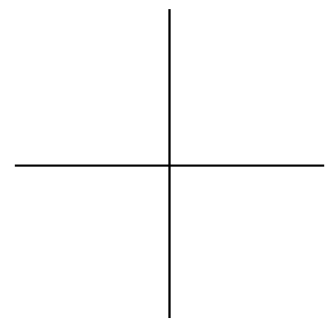
WORKSHEET ON COMPLEX NUMBERS

Task 1 $i^2 =$

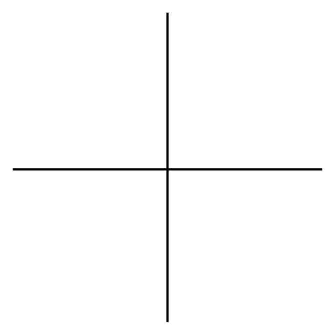
$(1 + i)(2 - 3i) =$

Task 2 Locate each of the following in the cartesian plane.

(a) 1, $(-1 + i\sqrt{3})/2$, $(-1 - i\sqrt{3})/2$



(b) 1, -1, i, $-i$



(c) Connect each of the points in (a) and describe the properties of the figure. Repeat with (b).

(d) What are the roots of $x^3 - 1 = 0$, $x^4 - 1 = 0$, and how is this question related to the above parts?

Task 3 If r and s are roots of $x^2 - 7x + 43 = 0$, what are $r + s$ and rs?

Task 4 Use $(x - a)(x - b) = x^2 - (a + b)x + ab$ to resolve Task 3.

Give a similar expression for $(x - a)(x - b)(x - c)$.

What is the sum of the roots of $x^3 - 3x^2 + 2x - 14 = 0$? The product of the roots?

Let 1, r, s be roots of $x^3 - 1 = 0$.

The product of the roots is $1rs =$____________, so that $rs =$____________.

Also, $r^3 =$____________, $s^3 =$____________.

Explain why $r^2 = -r - 1$ and $r = \frac{1}{s}$ and $r^3 = \frac{r^2}{s}$.

Why is $r^2 = s$?

MULTIPLICATION TABLE OF ROOTS

Task 1 Use $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$ to complete the chart:

	FACTORS	ROOTS
$x^2 - 1 = 0$	$(x-1)(x+1)$	$1, -1$
$x^3 - 1 = 0$		
$x^4 - 1 = 0$		

For convenience, you might want to label the roots of $x^3 - 1 = 0$ as 1, β, γ.

Task 2 Make the multiplication table for each set of roots:

	1	-1
1		
-1		

	1	β	γ
1			
β			
γ			

Task 3 Plot separately the set of roots for $x^2 - 1 = 0$, $x^3 - 1 = 0$, and $x^4 - 1 = 0$.

Connect the points and describe the geometrical figure produced.

Task 4 Conjecture what happens for $x^5 - 1 = 0$, $x^6 - 1 = 0$.

GROUP TABLE FOR THE SIXTH ROOTS OF UNITY

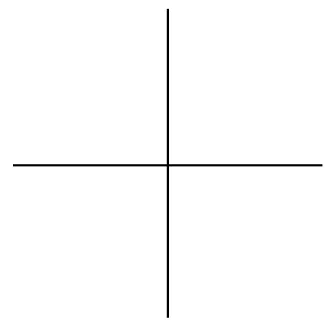
The goal here is to find the six roots of $x^6 - 1 = 0$, plot them, and make their group table.

Task 1 Factor $x^6 - 1 = (x^3 - 1)(\qquad) = (\qquad)(\qquad)(\qquad)(\qquad)$

The six roots are:

Task 2 Plot these six complex numbers.

Connect them, forming a ____________________



Task 3 Label the six roots 1, r, s, -1, $-r$, $-s$ where 1, r, s are the roots of $x^3 - 1 = 0$. Show why the last three are negatives of the first three. Show why $rs = 1$, $r^2 = s$, $s^2 = r$.

Task 4 Make the group table

	1	r	s	-1	$-r$	$-s$
1						
r						
s						
-1						
$-r$						
$-s$						

MULTIPLICATION TABLE FOR THE ROOTS OF $x^8 - 1 = 0$

FACTOR: $x^8 - 1 = (x^4 - 1)(x^4 + 1)$. The roots of $x^4 - 1 = 0$ are ________________

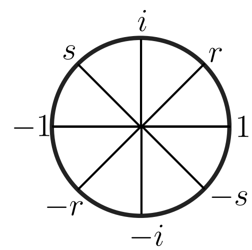
$$x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2 + 1)^2 - (\sqrt{2}\,x)^2 = (x^2 - \sqrt{2}\,x + 1)(x^2 + \sqrt{2}\,x + 1)$$

The other roots are:

$$r = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \qquad -r = \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$s = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \qquad -s = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

These eight roots are spread evenly around a unit circle 45° apart.



Task 1 Use the fact that when you multiply complex numbers you add their arguments to express each of the following in terms of $1, -1, i, -i, r, s, -r, -s$.

$r^2 =$ $\qquad$ $ir =$ $\qquad$ $s^2 =$

$rs =$ $\qquad$ $is =$

Task 2 Complete the multiplication table

	1	-1	i	$-i$	r	s	$-r$	$-s$
1								
-1								
i								
$-i$								
r								
s								
$-r$								
$-s$								

COMPOSITION OF FUNCTIONS

Let $f_1(x)=x$ $\quad f_4(x)=\dfrac{1}{x}$

$f_2(x)=\dfrac{1}{1-x}$ $\quad f_5(x)=1-x$

$f_3(x)=\dfrac{x-1}{x}$ $\quad f_6(x)=\dfrac{x}{x-1}$

The following "multiplication" table can be formed using composition of functions as the operation

EXAMPLE: $f_2 \circ f_6 = f_5$ since $f_2\left(\dfrac{x}{x-1}\right) = \dfrac{1}{1-\frac{x}{x-1}} = 1-x = f_5(x)$

$\circ$	x	$\frac{1}{1-x}$	$\frac{x-1}{x}$	$\frac{1}{x}$	$1-x$	$\frac{x}{x-1}$
$f_1 = x$	x	$\frac{1}{1-x}$	$\frac{x-1}{x}$	$\frac{1}{x}$	$1-x$	$\frac{x}{x-1}$
$f_2 = \frac{1}{1-x}$	$\frac{1}{1-x}$	$\frac{x-1}{x}$	x	$\frac{x}{x-1}$	$\frac{1}{x}$	$1-x$
$f_3 = \frac{x-1}{x}$	$\frac{x-1}{x}$					
$f_4 = \frac{1}{x}$	$\frac{1}{x}$					
$f_5 = 1-x$	$1-x$					
$f_6 = \frac{x}{x-1}$	$\frac{x}{x-1}$					

COMPOSITION OF FUNCTIONS (CONT)

Rewrite the table using $f_1, f_2, f_3, f_4, f_5, f_6$

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1						
f_2						
f_3						
f_4						
f_5						
f_6						

List properties of this table.

Complete the table showing inverses

f	f_1	f_2	f_3	f_4	f_5	f_6
f^{-1}						

List all the subsets of $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ that are closed under $\circ$.

EULER ϕ-FUNCTION

$\phi(n)$ is the number of positive integers less than n that are relatively prime to n. Here is a partial table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10					

Task 1 Complete the table. Any conjectures?

Task 2 Conjecture and prove a formula for $\phi(p)$, p a prime.

Task 3 Prove a formula for $\phi(p^2)$.

Task 4 Compute $\phi(7^3)$ by listing the integers.

Task 5 Prove a formula for $\phi(p^n)$.

Task 6 Prove that $\phi(11^n)$ is a multiple of 10, for all n.

Task 7 Show that $\phi(16) \cdot \phi(9) = \phi(16 \cdot 9)$

Task 8 Find all x such that $\phi(x) = n$ where:

(a) n =1 (b) n=2 (c) n=4

Task 9 The notation $(a, b) = 1$ means that a and b are relatively prime.
Prove that if $(a, m) = 1$, then $(m - a, m) = 1$.

Task 10 Prove that $\phi(n)$ is even for $n \geq 3$.

Task 11 It can be proved that if $(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$.
Use this to compute $\phi(72)$; also compute $\phi(120)$.

Task 12 Prove that if $n = p_1^{e_1} p_2^{e_2} p_3^{e_3}$, then $\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \left(1 - \frac{1}{p_3}\right)$.

WORKSHEET ON INVERTIBLES

Let $\mathbb{Z}_m = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \ldots, \overline{m-1})\}$ and V_m be the set of invertibles of $\mathbb{Z}_m$ consisting of those elements of $\mathbb{Z}_m$ that have *multiplicative* inverses. For each $\mathbb{Z}_m$ make a table of inverses of those elements that have multiplicative inverses and list V_m. Here the "bar" indicates an equivalence class. $\bar{2}$ indicates the set of all integers in $\mathbb{Z}$ whose remainder is 2 upon division by m. Once understood, the "bar" is omitted.

<u>SAMPLE</u>:

$\mathbb{Z}_3 = \{0, 1, 2\}$

x	0	1	2
x^{-1}		1	2

$V_3 = \{1, 2\}$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

x	0	1	2	3
x^{-1}				

$V_4 = \{ \qquad \}$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

x	0	1	2	3	4
x^{-1}					

$V_5 = \{ \qquad \}$

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

x	0	1	2	3	4	5
x^{-1}						

$V_6 = \{ \qquad \}$

$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

x	0	1	2	3	4	5	6
x^{-1}							

$V_7 = \{ \qquad \}$

Can you conjecture which elements of $\mathbb{Z}_{30}$ are invertibles?

How many invertibles does $\mathbb{Z}_p$ have where p is a prime?

How is the Euler ϕ-function related to these questions?

PRESERVATION OF OPERATION

In the following chart you are asked to verify whether certain familiar functions satisfy

$f(a \circ b) = f(a) \,\square\, f(b)$. The operations $\circ$ and $\square$ can be addition or multiplication or a mixture.

FUNCTION	YES OR NO	REASON
$f(x) = x^3$	yes	$f(xy) = (xy)^3 = x^3y^3 = f(x)f(y)$
$f(x) = x^4$		
$f(x) = e^x$		
$f(x) = \frac{3}{2}x$		
$f(x) = 2x + 1$		
$f(x) = \ln x$		
$f(x) = \|x\|$		
$f(x) = \sqrt{x}$		
$f(x) = 2x^3$		
$f(x) = \det x$		
$\theta(f) = f'$ (the derivative)		

A SPECIAL ISOMORPHISM

Task 1 Show that the mapping $\theta : G \to G$ given by $\theta(g) = g^{-1}$ is an isomorphism if G is abelian.

STEP 1 θ is 1-to-1. To show this we need to show that if $\theta(g_1) = \theta(g_2)$ then

____. Since $\theta(g_1) = \theta(g_2)$ we get

____. Now by taking inverses, we obtain

____.

STEP 2 Show that θ is onto.

STEP 3 Show that θ preserves the operation

Task 2 Use $\theta(g) = g^{-1}$ to tabulate an isomorphism from S_3, the group of symmetries of an equilateral triangle, to itself.

g	
$\theta(g)$	

Task 3 Show that the above result is false if G is not abelian.

MATCHING GROUPS

There are two nonisomorphic groups of order 4, the cyclic group and the Klein 4-group whose elements x satisfy $x^2 = e$.

For each of the following groups, label A if it is isomorphic to the cyclic group and B if it is isomorphic to the Klein group.

A. Cyclic
B. Klein

__________ [(1234)]

__________ $[-i]$

__________ $\{e, a, a^2, a^3\}$

__________ $[i]$

__________ Rectangle Group

__________ $\{1, -1, i, -i\}$

__________ $\{0, 1, 2, 3\}$ under addition mod 4

__________ $\{(1), (12), (34), (12)(34)\}$

WORKSHEET ON AN 8×8 GROUP TABLE – $\mathbb{Z}_8$

Task 1 Fill in the following table where each of the 64 entries is found by addition modulo 8; i.e. add the two numbers, divide by 8, and record the remainder.

$\oplus$	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

MAKE A TABLE OF INVERSES

DRAW THE LATTICE OF SUBGROUPS.

	0	1	2	3	4	5	6	7

LABEL AND LIST ALL THE SUBGROUPS.

WORKSHEET ON AN 8×8 GROUP TABLE – $\mathbb{Z}_2 \times \mathbb{Z}_4$

Task 1 Fill in the following table using bitwise addition mod 2 in the left-most slot and bitwise addition mod 4 in the right-most slot.

$\oplus$	00	01	02	03	10	11	12	13
00								
01								
02								
03								
10								
11								
12								
13								

MAKE A TABLE OF INVERSES

DRAW THE LATTICE OF SUBGROUPS.

LABEL AND LIST ALL THE SUBGROUPS.

WORKSHEET ON AN 8×8 GROUP TABLE – $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Task 1 Fill in the following table using bitwise addition mod 2.

$\oplus$	000	001	010	011	100	101	110	111
000								
001								
010								
011								
100								
101								
110								
111								

MAKE A TABLE OF INVERSES

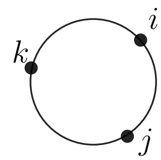
DRAW THE LATTICE OF SUBGROUPS.

LABEL AND LIST ALL THE SUBGROUPS.

WORKSHEET ON AN 8×8 GROUP TABLE – THE QUATERNIONS

Task 1 Fill in the following table using the operations:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$$



$\otimes$	1	-1	i	$-i$	j	$-j$	k	$-k$
1								
-1								
i								
$-i$								
j								
$-j$								
k								
$-k$								

MAKE A TABLE OF INVERSES

DRAW THE LATTICE OF SUBGROUPS.

LABEL AND LIST ALL THE SUBGROUPS.

WORKSHEET ON AN 8×8 GROUP TABLE – THE OCTIC GROUP

Task 1 Fill in the following table using $fr = r^3 f$. These eight elements are the eight symmetries of a square.

$\square$	1	r	r^2	r^3	f	rf	$r^2 f$	$r^3 f$
1								
r								
r^2								
r^3								
f								
rf								
$r^2 f$								
$r^3 f$								

MAKE A TABLE OF INVERSES

DRAW THE LATTICE OF SUBGROUPS.

LABEL AND LIST ALL THE SUBGROUPS.

FIVE NONISOMORPHIC GROUPS OF ORDER 8

Listed next are the elements of these five groups along with their names. You are asked to show why certain pairs are not isomorphic.

CYCLIC	$\{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$
QUATERNIONS	$\{1, -1, i, -i, j, -j, k, -k\}$
OCTIC	$\{(1), \rho, \rho^2, \rho^3, \phi, \rho\phi, \rho^2\phi, \rho^3\phi\}$
BIT STRINGS or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\{000, 001, 010, 011, 100, 101, 110, 111\}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\{00, 01, 02, 03, 10, 11, 12, 13\}$

Task 1 Give two reasons why the QUATERNIONS are not isomorphic to the OCTIC group.

Task 2 Why is $\mathbb{Z}_2 \times \mathbb{Z}_4$ not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$?

Task 3 Why is the CYCLIC group not isomorphic to any of the other four?

WORKSHEET ON GROUP TABLES, ISOMORPHISMS

Complete the following table using $\circ$ to mean composition of functions. For example.

$$f_2 \circ f_3 = f_2(f_3(x)) = f_2(1-x) = \frac{1}{1-x} = f_4$$

The six functions are:

$$f_1(x) = x \quad f_2(x) = \frac{1}{x} \quad f_3(x) = 1-x \quad f_4(x) = \frac{1}{1-x} \quad f_5(x) = \frac{x-1}{x} \quad f_6 = \frac{x}{x-1}$$

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1						
f_2						
f_3						
f_4						
f_5						
f_6						

Display an isomorphism between this group and either S_3 or $\mathbb{Z}_6$.

FUNDAMENTAL THEOREM OF FINITE ABELIAN GROUPS

THEOREM: Every finite abelian group can be written as a product of cyclic groups of prime power order.

EXAMPLES: The Klein 4–Group is $\mathbb{Z}_2 \times \mathbb{Z}_2$; the cyclic group of order 4 is $\mathbb{Z}_4$. The abelian group of order 6 is $\mathbb{Z}_6$ which is isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_3$.

Let $G = \{1, 8, 12, 14, 18, 21, 27, 31, 34, 38, 44, 47, 51, 53, 57, 64\}$ be a group of order 16 under multiplication mod 65. G is isomorphic to one of:

$$\mathbb{Z}_{16}$$
$$\mathbb{Z}_2 \times \mathbb{Z}_8$$
$$\mathbb{Z}_4 \times \mathbb{Z}_4$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

BUT WHICH ONE?

LOOK AT ORDERS!

x	1	8	12	14	18	21	27	31	34	38	44	47	51	53	57	64
order x	1	4	4	2		4	4			4	4	4		4	4	

Task 1 Why is G not $\mathbb{Z}_{16}$?

Task 2 Why is G not $\mathbb{Z}_2 \times \mathbb{Z}_8$?

Task 3 What are the orders of elements in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$?

Task 4 Which one must G be?

Task 5 $G = \{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group of order 9 under multiplication modulo 91. Is G isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$? Why are these the only two possibilities?

Make the table of orders and inverses.

x	1	9	16	22	29	53	74	79	81
Order x									

x	1	9	16	22	29	53	74	79	81
x^{-1}									

Task 6 Identify all abelian groups (up to isomorphism) of order 360 by doing the following:

A. Express 360 as a product of prime numbers

B. List the six direct product possibilities

WHICH DIRECT PRODUCT?

$V_{45} = \{1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28, 29, 31, 32, 34, 37, 38, 41, 43, 44\}$ is a multiplicative group, using mod 45, of order 24. According to the fundamental theorem of finite abelian groups, V_{45} is isomorphic to a direct product of cyclic groups, each having prime power order. The possibilities are:

$$\mathbb{Z}_3 \times \mathbb{Z}_8 \qquad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$$

You do not include $\mathbb{Z}_{24}$, $\mathbb{Z}_6 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ since the orders are not prime power. Also notice that the cyclic group $\mathbb{Z}_{24}$ is, in fact, $\mathbb{Z}_3 \times \mathbb{Z}_8$ which has (1, 1) as a generator.

One way of determining which of these three is isomorphic to V_{45} is by computing the orders of each element in V_{45} and comparing with orders of elements in the direct products. By hand, this is not an easy feat.

USING THE TI-92

Clear home screen – F1, 8
Clear input line – CLEAR
Type in

Define $f(n) = \text{mod}(22^n, 45)$

to determine the order of 22, eg., and enter.
Go to APPS and 6: Data/matrix editor, current and enter
If necessary clear columns with F6, 5.
Highlight C1 and enter. Generate the sequence 1, 2, . . . , 24 with

C1 = seq(n, n, 1, 24) and enter.

Highlight C2 and generate $f(n)$ with

C2 = seq($f(n)$, n, 1, 24) and enter.

Column C2 will give 22^n reduced modulo 45 for n ranging from 1 to 24. You should find that $22^{12} \equiv 1$.
Now, if you return to the home screen [2nd, QUIT] and just change the 22 to 2 we can quickly see, returning to APPS, 6, in column 2 that the order of 2 is 12. Repeat this and complete the following table of orders.

x	1	2	4	7	8	11	13	14	16	17	19	22
order x	1	12	6	12								

x	23	26	28	29	31	32	34	37	38	41	43	44
order x											12	2

RINGS THAT ARE NOT INTEGRAL DOMAINS

A commutative ring D with unity 1, having no zero-divisors is called an integral domain.

Task 1 Explain why $\mathbb{Z}_{10}$ is not an integral domain.

Task 2 Why is $\mathbb{Z}_{12}$ not?

Task 3 For which m is $\mathbb{Z}_m$ not an integral domain?

Task 4 Is $\mathbb{Z}_2 \times \mathbb{Z}_2$ an integral domain?

Task 5 Let A and B be integral domains. Is $A \times B$ an integral domain?

Task 6 Is the set of all 2×2 matrices with real entries with the usual addition an multiplication of matrices an integral domain?

WORKSHEET ON POLYNOMIALS IN $\mathbb{Z}_n[x]$

Task 1 Tabulate each of $(x+\bar{2})(x+\bar{5})$ and $x(x+\bar{7})$ in $\mathbb{Z}_{10}$ and thus show that they are equal.

x	0	1	2	3	4	5	6	7	8	9
$(x+\bar{2})(x+\bar{5})$			8							
$x(x+\bar{7})$			8							

Task 2 Show that $(x+\bar{3})(x+\bar{5}) = x(x+\bar{8})$ in $\mathbb{Z}_{15}[x]$.

Task 3 Show that $(x+\bar{6})^2 = x^2$ in $\mathbb{Z}_{12}[x]$.

Task 4 Our experience leads us to expect that deg $\alpha\beta$ = deg α + deg β

for polynomials α and β.

But ... Show that $(\bar{2}x+\bar{1})(\bar{3}x+\bar{5}) = x+\bar{5}$ in $\mathbb{Z}_6[x]$.

Task 5 Show that in $\mathbb{Z}_6[x]$

$$(\bar{2}x+\bar{5})(\bar{3}x+\bar{2}) = (x+\bar{4})$$

$$(\bar{3}x+\bar{3})(\bar{4}x^2+\bar{2}) = \bar{0}$$

Task 6 In $\mathbb{Z}_5[x], (\bar{2}x+\bar{1})(\bar{4}x+\bar{3}) = (x+\bar{3})(\bar{3}x+\bar{1})$

Make these linear factors monic (leading coefficient is $\bar{1}$) by factoring out $\bar{2}, \bar{4}$, and $\bar{3}$. For example $(\bar{2}x+\bar{1}) = \bar{2}(x+\bar{3})$. Then both sides become

________________________.

WORKSHEET ON ANOTHER RING

Define"addition" and "multiplication" as follows:

$$a \oplus b = a + b - 1$$

$$a \otimes b = a + b - ab$$

show that $(\mathbb{Z}, \oplus, \otimes)$ is a ring by doing the following:

Task 1 Determine the "0", the additive identity. Is there a unity?

Task 2 What is the additive inverse of a? Why?

Task 3 Is $\oplus$ associative? Is $\otimes$ associative?

Task 4 Show that $\otimes$ is distributive over $\oplus$.

SUMMARY OF RINGS AND THEIR PROPERTIES

Ring	Form of Element	Unity	Abelian	Integral Domain	Field
$\mathbb{Z}$	k	1	Yes	Yes	No
$\mathbb{Z}_n$, n composite					
$\mathbb{Z}_p$, p prime					
$\mathbb{Z}[x]$					
$n\mathbb{Z},\ n > 1$					
$M(\mathbb{Z})$, 2×2 matrices					
$\mathbb{Z}[i]$					
$\mathbb{Z}_3[i]$					
$\mathbb{Z}_2[i]$					
$\mathbb{Z}[\sqrt{2}]$					
$\mathbb{Q}[\sqrt{2}]$	$a + b\sqrt{2}$				
$\mathbb{Z} \oplus \mathbb{Z}$					

$\mathbb{Z}_3[i]$ IS A FIELD WITH 9 ELEMENTS

This field consists of all elements of the form $m + ni$ where m and n are in {0,1,2}. The multiplication table is:

	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
1	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
2	2	1	$2i$	$2+2i$	$1+2i$	i	$2+i$	$1+i$
i	i	$2i$						
$1+i$	$1+i$	$2+2i$						
$2+i$	$2+i$	$1+2i$						
$2i$	$2i$	i						
$1+2i$	$1+2i$	$2+i$						
$2+2i$	$2+2i$	$1+i$						

Task 1 Complete the table of inverses and orders:

x	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
x^{-1}								
order of x								

Task 2 Which familiar group is the multiplicative group isomorphic to?

APPLICATION OF A FAMOUS THEOREM

PROBLEM: Explain why $x^7 - x = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$ in $\mathbb{Z}_7[x]$.

There are a number of ways of approaching this problem, several motivated by a similar question you couild ask in the 8^{th} grade. Explain why $x^2 - 5x + 6 = (x-2)(x-3)$. The following tasks guide you through three solutions.

Task 1 Use a calculator and show that each of 1, 2, 3, 4, 5, 6 satisfies $x^7 - x = 0$; then use the factor theorem.

Task 2 Simplify the right hand side algebraically using $x - 6 = x + 1$, $x - 5 = x + 2$, $x - 4 = x + 3$.

Task 3 What famous Theorem has $x^7 - x \equiv 0$ or $x^7 \equiv x$ in it?